PROBLEM INVOLVING A THREE-DIMENSIONAL BOUNDARY LAYER IN A GENERALIZED STOKES FLUID

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A generalized Stokes fluid is defined, and a new similarity criterion, the Truesdell number, is discussed. Equations of the boundary-layer type in an incompressible gas are derived. A detailed study is made of the self-similar problem of a three-dimensional boundary layer.

1. Definition of a Generalized Stokes Fluid and Relation with the Kinetic Theory of Gases. Following [1], we define a generalized Stokes fluid as a continuous medium in which there exist certain material constants μ_0 and T_0, the natural viscosity and characteristic temperature, of dimensionality

$$[\mu_0] = M_m L^{-1} t^{-1}, \ [T_0] = \Theta,$$

and in which the viscous-stress tensor depends on the rate-of-strain tensor S_{ij} and on μ_0, T_0, p_m, p, and T:

$$\begin{gathered} \nu_{ij} = f(\mu_0, T_0, p_m, p, T, S_{ij}), \\ \nu_{ij} = 0, \quad \text{if} \quad S_{ij} = 0. \end{gathered} \tag{1}$$

For an isotropic medium, Eq. (1) becomes

$$\nu_{ij} = E_0\delta_{ij} + E_1 S_{ij} + E_2 S_{ik} S_{kj}, \tag{2}$$

where

$$E_0, \ E_1, \ E_2 = f(\mu_0, T_0, p_m, p, T, I_1, I_2, I_3). \tag{3}$$

Here I_1, I_2, and I_3 are the principal invariants of the rate-of-strain tensor. From the eight dimensional quantities formed from the three independent dimensionalities $M_m L^{-1} t^{-1}$, t, and Θ on the right side of Eq. (3), we can form the five dimensionless combinations

$$\frac{\mu_0}{p_0} I_1, \ \frac{\mu_0^2}{p_0^2} I_2, \ \frac{\mu_0^3}{p_0^3} I_3, \ \frac{p}{p_m}, \ \frac{T}{T_0}.$$

The dimensionless combinations on the left side of Eq. (3) are

$$F_0 = \frac{E_0}{p_0}, \quad F_1 = \frac{E_1}{\mu_0}, \quad F_2 = \frac{E_2 p_0}{\mu_0^2}.$$

We can then rewrite Eq. (3) in dimensionless form,

$$F_0, \ F_1, \ F_2 = f\left(\frac{\mu_0}{p_0} I_1, \ \frac{\mu_0^2}{p_0^2} I_2, \ \frac{\mu_0^3}{p_0^3} I_3, \ \frac{p}{p_m}, \ \frac{T}{T_0}\right),$$

and replace Eq. (2) by

$$\nu_{ij} = p_0 F_0 \delta_{ij} + \mu_0 F_1 S_{ij} + \frac{\mu_0^2}{p_0} F_2 S_{ik} S_{kj}. \tag{4}$$

Truesdell [1] proposed the following polynominal representation for the coefficients F_0, F_1, and F_2:

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$$F_\gamma = \left(\frac{\mu_0}{p_0} I_1\right)^l \left(\frac{\mu_0^2}{p_0^2} I_2\right)^m \left(\frac{\mu_0^3}{p_0^3} I_3\right)^n F_{\gamma lmn},$$

$$F_{\gamma lmn} = f\left(\frac{p}{p_m}, \frac{T}{T_0}\right), \quad F_{0000} = 0. \tag{5}$$

Within terms of second order inclusively, Eq. (4) becomes, with an account of (5),

$$\nu_{ij} = \mu_0 \left[F_{0100} I_1 \delta_{ij} + F_{1000} S_{ij}\right] + \frac{\mu_0^2}{p_0}\left[F_{0200} I_1^2 \delta_{ij} + F_{1100} I_1 S_{ij} + F_{0010} I_2 \delta_{ij} + F_{2000} S_{ik} S_{kj}\right]. \tag{6}$$

We introduce the shear-velocity tensor,

$$\mathring{S}_{ij} = S_{ij} - \frac{1}{3} I_1 \delta_{ij}, \quad \overline{\overline{S_{ik}S_{kj}}}^{\circ} = S_{ik}S_{kj} - \frac{1}{3}(I_1^2 - 2I_2)\, s_{ij}; \tag{7}$$

if

$$\begin{gathered} 3F_{0100} + F_{1000} = 0, \\ 3F_{0200} + F_{1100} + F_{2000} = 0, \\ 3F_{0010} - 2F_{2000} = 0, \end{gathered} \tag{8}$$

Eq. (6) becomes

$$\nu_{ij} = \mu_0 F_{1000} \mathring{S}_{ij} + \frac{\mu_0^2}{p_0}\left[F_{1100} I_1 \mathring{S}_{ij} + F_{2000} \overline{\overline{S_{ik}S_{kj}}}^{\circ}\right]. \tag{9}$$

The first relation in (8) is the Stokes condition in the classical case. We note that Eqs. (8) are the conditions for the equality of the pressure p to the hydrostatic pressure, obtained by Truesdell [1] in a different manner.

A new dimensionless complex $\mu_0 \mathring{S}_{ij}/p_0$, first introduced by Truesdell, appears among the nonlinear terms in Eq. (9); this complex is the criterion for the appearance of nonlinear effects. Accordingly, along with the ordinary Reynolds, Mach, etc. similarity criteria, we have a new one – the Truesdell number, which becomes significant at high altitudes or in shock waves, because of the sharp velocity gradient. We note that Predvoditelev [2] used the reciprocal of the Truesdell number in evaluating acoustic dispersion.

If the basic assumptions of the kinetic theory of gases hold, the Truesdell number can be related to the familiar Knudsen and Mach numbers:

$$\mathrm{Tr} = \frac{\mu_0 \mathring{S}_{ij}}{p_0} \sim \frac{l\rho\bar{c}\, U_m}{\rho\bar{c}^2 L_m} = \frac{l\, U_m}{L_m \bar{c}} \sim \mathrm{Kn}\,\mathrm{M} \sim \mathrm{M}^2 \mathrm{Re}^{-1}.$$

We note that we could derive Eq. (9) on the basis of molecular-kinetics concepts. For this purpose, we consider the expression for the viscous-stress tensor with an account of the Barnett approximation, which holds in the dynamics of low-density gases [3]:

$$\nu_{ij} = 2\mu \mathring{S}_{ij} + \omega_1 \frac{\mu^2}{p} I_1 \mathring{S}_{ij} + \frac{\omega_2 \mu^2}{p}\left[\frac{\partial}{\partial \mathbf{r}}\left(\mathbf{F} - \frac{1}{\rho}\frac{\partial p}{\partial \mathbf{r}}\right) - \left(\frac{\partial}{\partial \mathbf{r}}\mathbf{c}_0\right)\left(\frac{\partial}{\partial \mathbf{r}}\mathbf{c}_0\right) - 2\frac{\partial}{\partial \mathbf{r}}\mathbf{c}_0 \mathring{S}_{ij}\right] + \omega_3 \frac{\mu^2}{\rho T}\overline{\overline{\frac{\partial}{\partial \mathbf{r}}\frac{\partial T}{\partial \mathbf{r}}}}^{\circ}$$

$$+ \omega_4 \frac{\mu^2}{\rho p T}\overline{\overline{\frac{\partial p}{\partial \mathbf{r}}\frac{\partial T}{\partial \mathbf{r}}}}^{\circ} + \omega_5 \frac{\mu^2}{\rho T^2}\overline{\overline{\frac{\partial T}{\partial \mathbf{r}}\frac{\partial T}{\partial \mathbf{r}}}}^{\circ} + \omega_6 \frac{\mu^2}{p}\overline{\overline{\mathring{S}_{ij}\mathring{S}_{ij}}}^{\circ}. \tag{10}$$

In the expression in square brackets in the coefficient of ω_2, the last two terms show the dependence of the viscous-stress tensor on the antisymmetric part of the velocity-gradient tensor $(\partial/\partial \mathbf{r})\mathbf{c}_0$, i.e., on vortex motion $(1/2\,\mathrm{rot}\,\mathbf{c}_0)$. In deriving Eq. (9), however, we neglected this dependence, so in determining the nature of the coefficients in Eq. (9) we will neglect these two terms in Eq. (10). The other terms in the coefficient of ω_2 are on the order of the following dimensionless complex:

$$\frac{\mu^2}{p\rho L_m^2} \sim \frac{l^2\rho^2\bar{c}^2}{\rho^2\bar{c}^2 L_m^2} = \frac{l^2}{L_m^2}\,\mathrm{Kn}^2.$$

A similar complex can be formed from the coefficients of ω_3, ω_4, and ω_5, so their contributions are on the order of the square of the Knudsen number. The expressions in the coefficients of ω_1 and ω_6 are obviously on the order of the Truesdell number.

Since for supersonic flow of a continuous medium we can assume $\mathrm{Kn}^2 \ll \mathrm{Tr}$, we can replace (10) by

$$\nu_{ij} = 2\mu\mathring{S}_{ij} + \omega_1 \frac{\mu^2}{p} I_1\mathring{S}_{ij} + \omega_6 \frac{\mu^2}{p} \overline{\overline{\mathring{S}_{ij}\mathring{S}_{ij}}}. \qquad (11)$$

Comparing (9) and (11), we find

$$\mu_0 = 2\mu, \quad F_{1000} = 1, \quad F_{1100} = \frac{\omega_1}{4}, \quad F_{2000} = \frac{\omega_6}{4}, \quad p_0 = p, \qquad (12)$$

where, for Maxwell molecules [3], we have

$$\omega_1 = \frac{4}{3}\left(\frac{7}{2} - \frac{T}{\mu}\frac{d\mu}{dT}\right), \quad \omega_6 = 8. \qquad (13)$$

2. Derivation of the Equations of a Three-Dimensional Boundary Layer in an Incompressible Gas. For convenience, we rewrite Eq. (9) for an incompressible gas as

$$p_{ij} = -p\delta_{ij} + 2\mu\mathring{S}_{ij} + 4K_2\overline{\overline{\mathring{S}_{ik}\mathring{S}_{kj}}}, \qquad (14)$$

where

$$K_1 = \frac{\mu^2}{p_0} F_{1100}, \quad K_2 = \frac{\mu^2}{p_0} F_{2000}.$$

The general dynamical equations of a continuous medium with body forces neglected are [4]

$$\begin{aligned} \rho \frac{du}{dt} &= \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}, \\ \rho \frac{dv}{dt} &= \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z}, \\ \rho \frac{dw}{dt} &= \frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z}, \end{aligned} \qquad (15)$$

where the p_{ij} must be determined from Eq. (14). In system (15) we introduce the dimensionless variables

$$u = U\bar{u}, \quad v = V\bar{v}, \quad w = U\bar{w},$$
$$x = L\bar{x}, \quad y = \mathcal{Y}\bar{y}, \quad z = L\bar{z}, \quad p = P\bar{p}.$$

We set up the arbitrary scales as is done in the theory of a planar boundary layer [5]:

$$\mathcal{Y} = \frac{L}{\sqrt{\mathrm{Re}}}, \quad V = \frac{U}{\sqrt{\mathrm{Re}}}, \quad P = \rho U^2, \quad \mathrm{Re} = \frac{LU}{\nu}.$$

Then, discarding terms on the order of $1/\mathrm{Re}$ in (15), we find (omitting the bars over the dimensionless quantities)

$$\frac{du}{dt} = -\frac{\partial}{\partial x}\left(p - \frac{8}{3}AI_2\right) + \frac{\partial^2 u}{\partial y^2} + A\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2 + A\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\frac{\partial w}{\partial y} - 2\frac{\partial w}{\partial z}\frac{\partial u}{\partial y}\right] + A\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\frac{\partial w}{\partial y}\right),$$

$$0 = -\frac{\partial}{\partial y}\left(p - \frac{8}{3}AI_2\right) + A\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right],$$

$$\frac{dw}{dt} = -\frac{\partial}{\partial z}\left(p - \frac{8}{3}AI_2\right) + \frac{\partial^2 w}{\partial y^2} + A\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\frac{\partial w}{\partial y}\right) + A\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial z}\right) - 2\frac{\partial u}{\partial x}\frac{\partial w}{\partial y}\right] + A\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial y}\right)^2, \qquad (16)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

where

$$I_2 = -\frac{1}{4}\left[\left(\frac{\partial u}{\partial y}\right)^2 - \left(\frac{\partial w}{\partial y}\right)^2\right], \quad A = \mathrm{Tr}\,F_{2000}, \quad \mathrm{Tr} = \frac{\mu U}{p_0 L}.$$

The characteristic pressure p_0 can be chosen, e.g., equal to the pressure p_∞ in the incident flow.

TABLE 1. Components of the Friction Forces at the Wall for Certain Truesdell Numbers

Tr	a_1	b_1	$f''(0)$	$g'(0)$
0	0,2509	0,2509	1,2509	1,2509
0,15	0,0102	0,4335	1,0102	1,4335
0,50	−0,8287	0,7097	0,1713	1,7097

Integrating the second equation in system (16) over y, and assuming that $\partial u/\partial y$ and $\partial w/\partial y$ vanish at the outer boundary of the boundary layer, we find an equation for the pressure within this layer:

$$p = p_*(x,\ z) + \frac{1}{3} A\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]. \tag{17}$$

Here $p_*(x, z)$ is the specified pressure distribution at the outer boundary of the boundary layer. Substituting (17) into (16), we finally find

$$\begin{gathered}
\frac{du}{dt} = -\frac{\partial p_*}{\partial x} + \frac{\partial^2 u}{\partial y^2} + A\left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial y} - 2\frac{\partial w}{\partial z}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\frac{\partial w}{\partial y}\right) + \left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\,\partial y}\right)\right],\\
\frac{\partial w}{\partial t} = -\frac{\partial p_*}{\partial z} + \frac{\partial^2 w}{\partial y^2} + A\left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\frac{\partial w}{\partial x} - 2\frac{\partial u}{\partial x}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\frac{\partial w}{\partial y}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\,\partial z}\right)\right],\\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.
\end{gathered} \tag{18}$$

We can use the standard boundary conditions:

$$\begin{gathered}
u = v = 0 \quad \text{at} \quad y = 0,\\
u = U(x,\ z),\ w = W(x,\ z) \quad \text{at} \quad y = \infty,
\end{gathered} \tag{19}$$

where the external-flow velocities $U(x, z)$ and $W(x, z)$ must satisfy the Euler equations, given in our case by

$$\begin{gathered}
U\frac{\partial U}{\partial x} + W\frac{\partial U}{\partial z} = -\frac{\partial p_*}{\partial x},\\
U\frac{\partial W}{\partial x} + W\frac{\partial W}{\partial z} = -\frac{\partial p_*}{\partial z}.
\end{gathered} \tag{20}$$

For simplicity we assume that the velocities u, v, and w are functions of only the two variables x and y; then we can specify the dimensionless velocities $U(x)$ and $W(x)$ as

$$U(x) = x,\ W(x) = x,$$

converting system (18) to

$$u = xf'(y),\ v = -f(y),\ w = xg(y).$$

To determine f and g we must solve a system of nonlinear ordinary differential equations,

$$\begin{gathered}
f''' + ff'' + (1 - f'^2) + A(gg'' - g'^2) = 0,\\
g'' + fg' + (1 - f'g) + A[(gf'')' - 2f'g''] = 0
\end{gathered} \tag{21}$$

with numerical boundary conditions

$$\begin{gathered}
f(0) = f'(0) = g(0) = 0,\\
f'(\infty) = g(\infty) = 1.
\end{gathered} \tag{22}$$

Following Dorodnitsyn [6], we multiply all the terms in system (21) by the smoothing function $\varkappa(y)$ and integrate from zero to infinity; then we find a system of integrodifferential equations:

$$\begin{gathered}
\int_0^\infty \varkappa(y)\,[f''' + ff'' + (1 - f'^2) + A(gg'' - g'^2)]\,dy = 0,\\
\int_0^\infty \varkappa(y)\,\{g'' + fg' + (1 - f'g) + A[(gf'')' - 2f'g'']\}\,dy = 0.
\end{gathered} \tag{23}$$

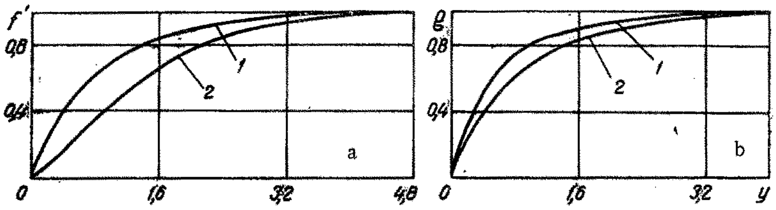

Fig. 1. Velocity components along the x axis (a) and along the z axis (b) in a three-dimensional boundary layer. 1) Tr = 0; 2) 0.5.

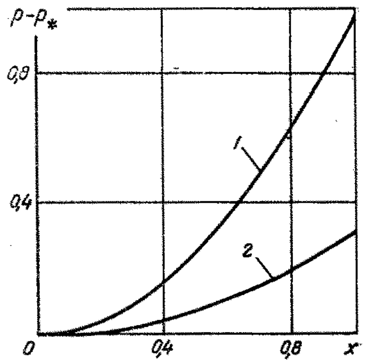

Fig. 2. Excess pressure arising at the wall in a three-dimensional boundary layer for certain Truesdell numbers. 1) Tr = 0.5; 2) 0.15.

Below we will use the first-approximation solution of system (23):

$$\begin{gathered} \varkappa_1(y) = e^{-y}, \\ f = \frac{a_1-2}{2} + y + (1-a_1)e^{-y} + \frac{a_1}{2}e^{-2y}, \\ f' = (1-e^{-y})(1+a_1e^{-y}), \\ g = (1-e^{-y})(1+b_1e^{-y}). \end{gathered} \tag{24}$$

Substituting (24) into (23), and integrating from zero to infinity, we find a first-approximation system of algebraic equations:

$$\begin{gathered} 71a_1 + 42Ab_1 = 18 - 36A - 3a_1^2 - 18Ab_1^2, \\ (3-12A)a_1 + (68-30A)b_1 = 18 + 25{,}2A + (18A-3)a_1b_1. \end{gathered} \tag{25}$$

The values of the friction at the wall in which we are interested are found in this approximation from

$$f''(0) = 1 + a_1, \quad g'(0) = 1 + b_1. \tag{26}$$

Table 1 shows the components of the friction force at the wall in the case of a Maxwell gas, calculated from Eqs. (26) for certain Truesdell numbers.*

From Fig. 1, which shows the velocity components within the boundary layer, we see that the profile of the component along the x axis approaches the separation profile with increasing Reynolds number, while the profile of the velocity component along the z axis becomes "fuller," and the boundary layer becomes narrower in this direction. Figure 2 shows the additional pressure calculated from Eq. (17).

NOTATION

ν_{ij}	is the viscous-stress tensor;
p_{ij}	is the stress tensor;
S_{ij}	is the rate-of-strain tensor;
$(\partial/\partial r)c_0$	is the velocity-gradient tensor;
$(\partial/\partial r)c_0 = \overset{\circ}{S}_{ij}$	is the rate-of-shear tensor;
Tr	is the Truesdell number;
Kn	is the Knudsen number;
M	is the Mach number;
M_m	is the dimensionality of mass;
L	is the dimensionality of length;
t	is the dimensionality of time;
Θ	is the dimensionality of temperature;
p_0	is the characteristic pressure in the flow;
l	is the molecular mean free path;
$\bar{c}$	is the arithmetic mean molecular velocity;
ρ	is the density;

*In this case the model of a Maxwell gas is not completely rigorous because of the approximate validity of Eq. (12).

U_m is the characteristic mass velocity;
L_m is the characteristic linear dimension of the solid;
Re is the Reynolds number;
u is the velocity component along the x axis;
v is the velocity component along the y axis;
w is the velocity component along the z axis.

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